JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **88**, 505–507 (1982)

On the Bellman Generalization of Steffensen's Inequality

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Steffensen [1] proved the following result: Assume that two integrable functions $f(t)$ and $g(t)$ are defined on the interval (a, b), that $f(t)$ never increases, and that $0 \leqslant g(t) \leqslant 1$ in (a, b). Then

$$\int_{b-\lambda}^{b} f(t)\,dt \leqslant \int_{a}^{b} f(t)\,g(t)\,dt \leqslant \int_{a}^{a+\lambda} f(t)\,dt \qquad \left(\lambda = \int_{a}^{b} g(t)\,dt\right). \tag{1}$$

Bellman [2] gives a new proof of Steffensen's inequality (1) and the following generalization (see also [3, p. 41; 4, p. 11]):

Let $f(t)$ be a nonnegative and monotone decreasing function in $[a, b]$ and $f \in L^p[a, b]$ and let $g(t) \geqslant 0$ in $[a, b]$ and $\int_a^b g(t)^q\,dt \leqslant 1$, where $p > 1$ and $(1/p) + (1/q) = 1$. Then

$$\left(\int_{a}^{b} f(t)\,g(t)\,dt\right)^{p} \leqslant \int_{a}^{a+\lambda} f(t)^p\,dt \qquad \left(\lambda = \left(\int_{a}^{b} g(t)\,dt\right)^{p}\right). \tag{2}$$

Bellman's result, however, is incorrect as has been noted by Godunova and Levin [5], where a generalization in a different sense is made ($p \leqslant 1$). An inequality for $p \geqslant 1$, similar to inequality (2), is given in [6].

In this paper, we shall show that the Bellman generalization of Steffensen's inequality, with very simple modifications of conditions, is true.

THEOREM 1. *Let f: $[0, 1] \to R$ be a nonnegative and nonincreasing function and let g: $[0, 1] \to R$ be an integrable function such that $0 \leqslant g(x) \leqslant 1$ $(\forall x \in [0, 1])$. If $p \geqslant 1$, then*

$$\left(\int_{0}^{1} g(t)\,f(t)\,dt\right)^{p} \leqslant \int_{0}^{\lambda} f(t)^p\,dt \tag{3}$$

where

$$\lambda = \left(\int_{0}^{1} g(t)\,dt\right)^{p}. \tag{4}$$

0022-247X/82/080505-03$02.00/0

Proof. Using the Jensen inequality for convex function $\Phi(x) = x^p$ $(p \geqslant 1)$, we have

$$\left(\int_0^1 g(t)\, f(t)\, dt\right)^p \leqslant \left(\int_0^1 g(t)\, dt\right)^{p-1} \int_0^1 g(t)\, f(t)^p\, dt.$$

To complete the proof we must prove

$$\left(\int_0^1 g(t)\, dt\right)^{p-1} \int_0^1 g(t)\, f(t)^p\, dt \leqslant \int_0^\lambda f(t)^p\, dt.$$

This inequality may be derived as follows:

$$\begin{aligned}
&\int_0^\lambda f(t)^p\, dt - \left(\int_0^1 g(t)\, dt\right)^{p-1} \int_0^1 f(t)^p\, g(t)\, dt \\
&\quad = \int_0^\lambda f(t)^p \left(1 - g(t) \left(\int_0^1 g(s)\, ds\right)^{p-1}\right) dt \\
&\qquad - \left(\int_0^1 g(s)\, ds\right)^{p-1} \int_\lambda^1 f(t)^p\, g(t)\, dt \\
&\quad \geqslant f(\lambda)^p \int_0^\lambda \left(1 - g(t) \left(\int_0^1 g(s)\, ds\right)^{p-1}\right) dt \\
&\qquad - \left(\int_0^1 g(s)\, ds\right)^{p-1} \int_\lambda^1 g(t)\, f(t)^p\, dt \\
&\quad = f(\lambda)^p \left(\lambda - \left(\int_0^1 g(s)\, ds\right)^{p-1} \int_0^\lambda g(t)\, dt\right) \\
&\qquad - \left(\int_0^1 g(s) ds\right)^{p-1} \int_\lambda^1 g(t)\, f(t)^p\, dt \\
&\quad = f(\lambda)^p \left(\left(\int_0^1 g(s)\, ds\right)^p - \left(\int_0^1 g(s)\, ds\right)^{p-1} \int_0^\lambda g(t)\, dt\right) \\
&\qquad - \left(\int_0^1 g(s)\, ds\right)^{p-1} \int_\lambda^1 g(t)\, f(t)^p\, dt \\
&\quad = \left(\int_0^1 g(s)\, ds\right)^{p-1} \left(f(\lambda)^p \int_\lambda^1 g(t)\, dt - \int_\lambda^1 g(t)\, f(t)^p\, dt\right) \\
&\quad = \left(\int_0^1 g(s)\, ds\right)^{p-1} \int_\lambda^1 g(t)(f(\lambda)^p - f(t)^p)\, dt \\
&\quad \geqslant 0.
\end{aligned}$$

Analogously, we can prove

THEOREM 2. *Let f: $[0, 1] \to R$ be a nonincreasing function and let g: $[0, 1] \to R$ be an integrable function such that $0 \leqslant g(x) \leqslant 1$ $(\forall x \in [0, 1])$. If $p \geqslant 1$, then*

$$\int_0^1 g(t) f(t)\, dt \Big/ \int_0^1 g(t)\, dt \leqslant \frac{1}{\lambda} \int_0^\lambda f(t)\, dt,$$

where λ is given by (4).

For $p = 1$, we have Steffensen's inequality.

Remark. If the functions f and g are defined on $[a, b]$, using the substitution $x = (b - a)\, t + a$, we can obtain the corresponding results.

ACKNOWLEDGMENTS

The author is grateful to Professor E. F. Beckenbach for useful suggestions.

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